Lecture on rewriting theory

# Homework

The homework must be written in English.
All answers must be clearly justified.

---

**Exercise 1 (1 point)** Let $A$ and $B$ be two sets, $R$ and $E$ be two relations on $A$, and $S$ be a relation on $B$. Then, let $L$ be the relation on $A \times B$ such that $(a,b) L_{R,E,S} (a',b')$ iff $aRa'$ or else $aEa'$ and $bSb'$. Prove that $L_{R,E,S}$ terminates if $R$ terminates, $RE \subseteq R$ and $S$ terminates.

---

We consider a set $\mathcal{F}$ of function symbols of fixed arity and an infinite set $\mathcal{V}$ of variables. Let $\mathcal{T}$ be the set of terms one can built from $\mathcal{F}$ and $\mathcal{V}$.

---

**Exercise 2 (1 point)** Prove that the rewrite system $f(gx) \to g(f(fx))$ cannot be proved terminating by using a polynomial interpretation on $\mathbb{N}$.

---

Let Rel be the set of relations on $\mathcal{T}$.
Let $\mathcal{S}$ be a function on Rel with the following properties:

(S1) If $\{R_k\}_{k \in K}$ is a totally ordered subset of Rel wrt $\subseteq$, then $\mathcal{S}(\bigcup_{k \in K} R_k) = \bigcup_{k \in K} \mathcal{S}(R_k)$.

(S2) If $R$ is a strict order (transitive and irreflexive), then $\mathcal{S}(R)$ is a strict order.

(S3) If $R$ is stable (by substitution), then $\mathcal{S}(R)$ is stable.

(S4) If $tRu$ then, for all symbols $f$ and terms $\vec{a}$ and $\vec{b}$, $f\vec{a}t\vec{b}\ \mathcal{S}(R)\ f\vec{a}u\vec{b}$.

(S5) If $t_0 \mathcal{S}(R) t_1 \mathcal{S}(R) \ldots$ is an infinite sequence, then there are $k$ and an infinite sequence $u_k R u_{k+1} R \ldots$ such that $u_k$ is a strict subterm of $t_k$.

---

**Exercise 3 (1 point)** Prove that $\mathcal{S}$ is monotone wrt inclusion.

---

**Exercise 4 (4 points)** Assume that the arity of function symbols is bounded. Let $\mathcal{L} : \text{Rel} \to \text{Rel}$ be the function such that, for all $R \in \text{Rel}$, $t\ \mathcal{L}(R)\ u$ iff there are $f$, $\vec{t}$, $g$, $\vec{u}$ and $i$ such that $t = f\vec{t}$, $u = g\vec{u}$, $t_i R u_i$ and, for all $j < i$, $t_j = u_j$. Prove that $\mathcal{L}$ satisfies the conditions (S1)-(S5).

---

A partially ordered set (poset) $(X, \leq)$ is strictly inductive if every non-empty totally ordered subset has a least upper bound in $X$.

**Fixpoint theorem (admitted):** In a strictly inductive poset having a smallest element, every monotone function has a least fixpoint.

**Induction principle (admitted):** Let $(A, \leq)$ be a strictly inductive poset with a smallest element $\perp$, $f$ be a monotone function on $A$, $a$ be the least fixpoint of $f$, and $P$ be a subset of $A$. Then, $a \in P$ if $f(P) \subseteq P$ and $(P, \leq)$ is a strictly inductive poset with $\perp$ as smallest element.

---

A quasi-ordering is a relation that is reflexive and transitive. Given a quasi-ordering $\succeq$, let $\succ = \succeq \setminus \preceq$ be its strict part, and $\simeq = \succeq \cap \preceq$ be its associated equivalence relation. A quasi-ordering $\succeq$ is said to be well-founded if $\succ$ terminates.

---

Let $\succeq_{\mathcal{F}}$ be a well-founded quasi-ordering on $\mathcal{F}$.

Let $\succeq_1$ be the reflexive closure of $\succ_1$, and $\succ_1$ be the smallest relation on $\mathcal{T}$ such that $t \succ_1 u$ iff there are $f$ and $\vec{t}$ such that $t = f\vec{t}$ and either:

(R1) there is $i$ such that $t_i \succeq_1 u$;

(R2) there are $g$ and $\vec{u}$ such that $u = g\vec{u}$, $f \succ_{\mathcal{F}} g$ and, for all $i$, $t \succ_1 u_i$;

(R3) there are $g$ and $\vec{u}$ such that $u = g\vec{u}$, $f \simeq_{\mathcal{F}} g$, $t\mathcal{S}(\succ_1)u$ and, for all $i$, $t \succ_1 u_i$.

Note that, when $\mathcal{S} = \mathcal{L}$, $\succ_1$ is the lexicographic path ordering (LPO) $\succ_{lpo}$.

---

**Exercise 5 (6 points)** Prove that $\succ_1$ is:

(a) well defined

(b) well-founded

(c) irreflexive

(d) stable (by substitution)

(e) monotone

(f) transitive if $t\mathcal{S}(\succ_1)u\mathcal{S}(\succ_1)v \Rightarrow t\mathcal{S}(\succ_1)v$ whenever, for all $(t', u', v')(\lhd, \lhd, \lhd)_{\text{lex}}(t, u, v)$, $t' \succ_1 u' \succ_1 v' \Rightarrow t' \succ_1 v'$

---

**Exercise 6 (1 point)** Prove that the rewrite system $f(gx) \to g(f(fx))$ cannot be proved terminating by LPO.