Rewriting techniques: Bonus

10–22–2025

Exercise 1 :
Prove the termination of the following TRS by finding a polynomial interpretation on integers:

$$x \times (y + z) \to (x \times y) + (x \times z)$$
$$(x + y) + z \to x + (y + z)$$

Exercise 2 :

(a) Find r_1 and r_2 such that $\{\mathsf{f}\ (\mathsf{g}\ x) \to r_1, \mathsf{g}\ (\mathsf{h}\ x) \to r_2\}$ is confluent.

(b) Show that the following string rewrite system is convergent

$$\mathsf{f}\ \mathsf{f} \to \mathsf{f} \qquad \mathsf{f}\ \mathsf{g} \to \mathsf{g}$$
$$\mathsf{g}\ \mathsf{g} \to \mathsf{f} \qquad \mathsf{g}\ \mathsf{f} \to \mathsf{g}$$

Can you determine the normal form of a term as a function of the numbers of fs and gs in it?

Exercise 3 :
In the following, we refer to s, t and w as in the definition of KBO. We will now prove some properties of this order to make the definition of KBO more clear.

(a) Assume that f is of arity 1, $w(f) = 0$ and that there is g such that $f \not> g$. Prove that under this conditions $>_{\text{kbo}}$ does not satisfy the subterm property.

Prove that, if w is admissible w.r.t. the strict order $>$ then $>_{\text{kbo}}$ on $T(\Sigma, V)$ induced by $>$ and w has the subterm property. To do so, prove the followings:

(b) Assume that $w(s) = w(t)$ and that t is a strict subterm of s. Prove that there exist a unary function f and a positive integer k such that $w(f) = 0$ and $s = f^k(t)$;

(c) Prove that $>_{\text{kbo}}$ is a strict order;

(d) Prove that $>_{\text{kbo}}$ is a rewrite order;

(e) Conclude that $>_{\text{kbo}}$ has the subterm property.

Exercise 4 :
Consider binary positive numbers defined thanks to the following three constructors:

- h : pos
- I : $\mathsf{pos} \to \mathsf{pos}$
- O : $\mathsf{pos} \to \mathsf{pos}$

(a) Write a TRS on a typed signature including a binary symbol + computing the addition of two natural numbers in this representation.

(b) Prove termination and confluence of this TRS.

(c) Describe closed normal forms of type pos

(d) Give a model in $\mathbb{N}$ of this TRS such that the function + is indeed represented by addition.

(e) Let $r : \mathbb{N}^* \to \mathcal{T}$ be the representation of positive integers using the constructors above.
Prove the correctness of this TRS: for all positive integers n and m, the normal form of $r(n) + r(m)$ is $r(n + m)$.

Exercise 5 :
Encode a non-deterministic turing machine as a TRS.