Rewriting Techniques, 4: dependency pairs, argument filtering

2025-10-08

Exercise 1 :
Is the TRS consisting of the rewriting rules

$$\begin{array}{lll} 0 + x \to x & \mathsf{gcd}\ x\ 0 \to x & \mathsf{gcd}\ (x+y)\ x \to \mathsf{gcd}\ x\ y \\ \mathsf{s}\ x + y \to \mathsf{s}\ (x+y) & \mathsf{gcd}\ 0\ x \to x & \mathsf{gcd}\ x\ (x+y) \to \mathsf{gcd}\ x\ y \end{array}$$

confluent?

Exercise 2 :
Compute the critical pairs of the following rewrite systems. Which one are locally confluent?

(a) $\mathsf{s}(\mathsf{p}(\mathsf{s}(y))) \to y$, $\mathsf{s}(\mathsf{p}(x)) \to \mathsf{p}(\mathsf{s}(x))$

(b) $0 + y \to y$, $x + 0 \to x$, $\mathsf{s}(w) + z \to \mathsf{s}(w+z)$, $v + \mathsf{s}(k) \to \mathsf{s}(v+k)$

(c) $\mathsf{a}(x,x) \to 0$, $\mathsf{a}(y,\mathsf{p}(y)) \to 1$

(d) $\mathsf{a}(\mathsf{a}(x,y),z) \to \mathsf{a}(x,\mathsf{a}(y,z))$, $\mathsf{a}(w,1) \to w$

$(\mathcal{A}, >, \gtrsim)$ is a weakly monotone algebra if $(\mathcal{A}, \gtrsim)$ is a monotone $\mathcal{F}$-algebra, $\gtrsim$ is a pre-order and $>$ is an order such that and $>\gtrsim \subset \gtrsim$ or $\gtrsim > \subset \gtrsim$. it is well-founded if $>$ so is.

Exercise 3 :
Prove that if $(\mathcal{A}, >, \gtrsim)$ is a well-founded weakly monotone algebra then $(>_{\mathcal{A}}, \gtrsim_{\mathcal{A}})$ is a reduction pair.

An argument filtering for a signature $\mathcal{F}$ is a mapping π that associates with every n-ary function symbol an argument position $i \in \{1, \cdots, n\}$ or a (possibly empty) list $[i_1, \cdots, i_m]$ of argument positions with $1 \leq i_1 < \cdots < i_m \leq n$.
The signature $\mathcal{F}_\pi$ consists of all function symbols f such that $\pi(f) = [i_1, \cdots, i_m]$, where in $\mathcal{F}_\pi$ the arity of f is m.
Every argument filtering π induces a mapping from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}_\pi, \mathcal{V})$, also denoted by π, which is defined as:

- $\pi(x) = x$ for all $x \in \mathcal{V}$
- $\pi(ft_1 \cdots t_n) = \pi(t_i)$ if $\pi(f) = i$
- $\pi(ft_1 \cdots t_n) = f(\pi(t_{i_1})) \cdots (\pi(t_{i_m}))$ if $\pi(f) = [i_1, \cdots, i_m]$.

Then for any relation R on $\mathcal{T}(\mathcal{F}, \mathcal{V})$, define $\pi(R) := \{(\pi(l), \pi(r)), (l,r) \in R\}$.

Exercise 4 :
Prove that a DP-problem (R, P) terminates if $(\pi(R), \pi(P))$ terminates

Exercise 5 :
Let R be the following TRS,

$$\mathsf{0}{-}y \to \mathsf{0} \tag{2}$$
$$x{-}\mathsf{0} \to x \tag{3}$$
$$\mathsf{s}\ x{-}\mathsf{s}\ y \to x{-}y \tag{4}$$
$$\mathsf{0}{\div}\mathsf{s}\ y \to \mathsf{0} \tag{5}$$
$$\mathsf{s}\ x{\div}\mathsf{s}\ y \to \mathsf{s}\ ((x{-}y){\div}\mathsf{s}\ y) \tag{6}$$

(a) Give dependency pairs of R

(b) Find a weakly monotone polynomial interpretation on $\mathbb{N}$ to prove termination of R.

(c) Find an argument filter to prove termination using LPO with empty precedence.

Exercise 6 :
Consider the rewriting system R:

$$\begin{array}{ccc}
\mathsf{0} \leq y \to \mathsf{true} & x - \mathsf{0} \to x & \mathsf{gcd}\ \mathsf{0}\ y \to y \\
(\mathsf{s}\ x) \leq \mathsf{0} \to \mathsf{false} & x - (\mathsf{s}\ y) \to \mathsf{p}\ (x{-}y) & \mathsf{gcd}\ (\mathsf{s}\ x)\ 0 \to \mathsf{s}\ x \\
\mathsf{s}\ x \leq \mathsf{s}\ y \to x \leq y & \mathsf{p}\ (\mathsf{s}\ x) \to x & \mathsf{gcd}\ (\mathsf{s}\ x)\ (\mathsf{s}\ y) \to \mathsf{if}\ (y \leq x)\ (\mathsf{s}\ x)\ (\mathsf{s}\ y) \\
\mathsf{if}\ \mathsf{true}\ (\mathsf{s}\ x)\ (\mathsf{s}\ y) \to \mathsf{gcd}\ (x{-}y)\ (\mathsf{s}\ y) & \mathsf{if}\ \mathsf{false}\ (\mathsf{s}\ x)\ (\mathsf{s}\ y) \to \mathsf{gcd}\ (y - x)\ (\mathsf{s}\ x) &
\end{array}$$

(a) Compute the dependency pairs of R.

(b) How many different argument filters does $R \cup \mathrm{DP}(R)$ admit?

(c) Prove the termination of R.

Exercise 7 :
Complete the ES consisting of the equation $(x \cdot y) \cdot (y \cdot z) \approx y$ (of central groupoids).