Rewriting Techniques, 2: termination, interpretation

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Exercise 1 :
Show that the strict order $>$ defined by

$$s > t \text{ iff } |s| > |t| \text{ and, for all } x \in FV(t), |s|_x > |t|_x$$

is a reduction order, where $|s|$ is the *size* of s and $|s|_x$ is the number of occurrences of x in s.

A **matrix interpretation on integers** is the following:

- a positive integer d;
- for every symbol f of arity n, n matrices $M_{f,1} \ldots, M_{f,n} \in \mathbb{N}^{d \times d}$;
- for every symbol of arity n, a vector $V_f \in \mathbb{N}^d$;
- a non-empty set $I \subseteq \{1, \ldots, d\}$ satisfying that for every symbol f of arity n the map

  $$L_f : (\mathbb{N}^d)^n \to \mathbb{N}^d \text{ defined as } L_f(X_1, \ldots, X_n) = V_f + \sum_{i=1}^{n} M_{f,i} X_i$$

  is monotonic with respect to $>_I$ were $X >_I Y$ holds if and only if for every $i \in \{1, \ldots, d\}$, $X[i] \geq Y[i]$ and there is $j \in I$ such that $X[j] > Y[j]$.

Then $(\mathbb{N}^d, (L_f)_f, >_I)$ is a well-founded monotone algebra.

Exercise 2 :
Consider the TRS $\{\ \mathsf{s}(\mathsf{a}) \to \mathsf{s}(\mathsf{p}(\mathsf{a})),\ \mathsf{p}(\mathsf{b}) \to \mathsf{p}(\mathsf{s}(\mathsf{b}))\ \}$.

1. Prove that its termination cannot be proved by a polynomial interpretation on integers;
2. Use the following matrix interpretation to prove termination w.r.t. $>_{\{1,2\}}$.

$$L_{\mathsf{s}}(X) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} X \qquad L_{\mathsf{p}}(X) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} X \qquad L_{\mathsf{a}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad L_{\mathsf{b}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

3. Why does it fail if we take $>_{\{1\}}$ instead? Is there another matrix interpretation that works with this ordering?

Exercise 3 :
Consider exercise on semantic labeling of TD1, where the following rewriting system was defined

$$\mathsf{f}\ (\mathsf{s}\ X) \to \mathsf{f}\ (\mathsf{p}\ (\mathsf{s}\ X)) \diamond (\mathsf{s}\ X) \qquad\qquad \mathsf{p}\ (\mathsf{s}\ \mathsf{z}) \to \mathsf{z}$$
$$\mathsf{p}\ (\mathsf{s}\ (\mathsf{s}\ X)) \to \mathsf{s}\ (\mathsf{p}\ (\mathsf{s}\ X))$$

1. Prove that RPO cannot prove the termination of the system.
2. Prove that a labeled system can be proved terminating with RPO.

Exercise 4 :
We consider the Ackermann's function

$$\begin{aligned} &\mathsf{Ack}\ 0\ y = y + 1 \\ &\mathsf{Ack}\ x\ 0 = \mathsf{Ack}\ (x-1)\ 1 \\ &\mathsf{Ack}\ x\ y = \mathsf{Ack}\ (x-1)\ (\mathsf{Ack}\ x\ (y-1)) \end{aligned}$$

(a) Prove its termination by induction.

(b) The following rewrite system simulates Ack

$$\mathsf{a}(\mathsf{0}, y) \to \mathsf{s}(y)$$
$$\mathsf{a}(\mathsf{s}(x), \mathsf{0}) \to \mathsf{a}(x, \mathsf{s}(\mathsf{0}))$$
$$\mathsf{a}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{a}(x, \mathsf{a}(\mathsf{s}(x), y))$$

Prove its termination using a LPO.

(c) Consider the well-founded domain $(\mathsf{Mult}(\mathbb{N} \times \mathbb{N}), (>_{\text{lex}})_{\text{mul}})$. Prove the termination of Ack using the following abstraction:

$$\phi : T(\{\mathsf{a}, \mathsf{s}\}, X) \to \mathsf{Mult}(\mathbb{N} \times \mathbb{N})$$
$$t \to \{\!\!\{ (|u|, |v|) \mid t|_{p \in \text{Pos}(t)} = \mathsf{a}(u, v) \}\!\!\}$$

where $|\mathsf{0}| = 1$, $|\mathsf{a}(x, y)| = |x| + |y| + 1$ and $|\mathsf{s}(x)| = |x| + 1$.

Exercise 5 :
Let $A \subseteq \mathbb{N}$ and P_f be respectively the domain and the interpretation, for each function symbol f, of a polynomial interpretation of integers for a TRS *(note: the TRS is therefore terminating)*. Take $a \in A \setminus \{0\}$.

1. Define $\pi_a : T(F, X) \to A$ as the function which maps every variable x to a and every term of the form $f(t_1, \ldots, t_n)$ to $\mathsf{P}_f(\pi_a(t_1), \ldots, \pi_a(t_n))$. Prove that $\pi_a(t)$ is greater or equal to the length of every reduction starting from t.
2. for every $n \in \mathbb{N}$ and $f \in F_n$, Show that there exists d_f and k_f positive integers such that for every $a_1, \ldots, a_n \in A \setminus \{0\}$, $\mathsf{P}_f(a_1, \ldots, a_n) \leq d_f \prod_{i=1}^{n} a_i^{k_f}$.
3. From the previous point, pick d_f to be also greater or equal than a for all f and fix $c_f \geq k_f + \log_2(d_f)$. For every $t \in T(F, X)$ let $c_t = \max_{f \in F_t} c_f$ where F_t is the finite set of all functions symbols appearing in t, then prove that $\pi_a(t) \leq 2^{2^{c_t |t|}}$.

Consider now any finite TRS R and a function symbol f. Prove that there exists an integer k such that if $s \to_R t$ then $|t|_f \leq k(|s|_f + 1)$, where $|.|_f$ is the number of f.
Deduce that the TRS

$$\{\ \mathsf{a}(\mathsf{0}, y) \to \mathsf{s}(y),\ \mathsf{a}(\mathsf{s}(x), \mathsf{0}) \to \mathsf{a}(x, \mathsf{s}(\mathsf{0})),\ \mathsf{a}(\mathsf{s}(x), \mathsf{s}(y)) \to \mathsf{a}(x, \mathsf{a}(\mathsf{s}(x), y))\ \},$$

simulating the Ackermann's function, cannot be proved terminating using a polynomial interpretation over integers.

A weight function for a signature Σ is a pair (w, w_0) consisting of a mapping $w : \Sigma \to \mathbb{N}$ and a constant $w_0 > 0$ such that $w(c) \geq w_0$ for all *constant* $c \in \Sigma$. Let (w, w_0) be a weight function. The weight of a term t is defined as follows

$$w(t) = \begin{cases} w_0 & \text{if } t \text{ is a variable} \\ w(f) + \sum_{i=1}^{n} w(t_i) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

We denote $|s|_x$ for x a variable the number of times that x occurs in s. Let $>$ be a precedence and (w, w_0) a weight function. We define the *Knuth-Bendix order* (KBO) $>_{\text{kbo}}$ on terms inductively as follows: $s >_{\text{kbo}} t$ if $|s|_x \geq |t|_x$ for all variables x and either

1. $w(s) > w(t)$, or
2. $w(s) = w(t)$ and one of the following alternatives holds
   (a) t is a variable and $s = f^n(t)$ for some unary function symbol f and $n > 0$,
   (b) $s = f(s_1, \ldots, s_n), t = f(t_1, \ldots, t_n)$ and there is $i \in \{1, \ldots, n\}$ such that $s_j = t_j$ for all $1 \leq j < i$ and $s_i >_{\text{kbo}} t_i$, or
   (c) $s = f(s_1, \ldots, s_n), t = g(t_1, \ldots, t_m)$ and $f > g$.

Exercise 6 :
Using a KBO, prove the termination of:

1. $\{\mathsf{I}(x) + (y + z) \to x + (\mathsf{I}(\mathsf{I}(y)) + z),\ \mathsf{I}(x) + (y + (z + w)) \to x + (z + (y + w))\ \}$
2. $\{\ \mathsf{r}^n(\mathsf{I}^k(x)) \to \mathsf{I}^k(\mathsf{r}^m(x))\ \}$, where $n, k > 0$ and $m \geq 0$.