Rewriting techniques, I: basics, interpretation, termination

2025–09–10

Exercise 1 :

(a) Given the reduction rules

$$((s\ x) + y) \rhd (s\ (x + y)); \quad (0 + x) \rhd x$$

Can $(s\ ((s\ 0) + 0))$ be *reduced*? Can it be *rewritten*? Provide the substitution, the context and the term t being reduced.

(b) A *string rewrite system* (SRS for short) is a TRS over a signature that contains only unary function symbols. Given the (string) reductions

$$a(b(x)) \rhd b(a(x))$$

Can $a(a(b(x)))$ be reduced? Can $a(b(a(x)))$ be reduced? Can they be rewritten?

(c) Build a reduced string rewrite system that is not terminating.

Solution:
Definition of an interreduced TRS from
https://www.lix.polytechnique.fr/ jouannaud/articles/cours-tlpo.pdf:
A TRS R is interreduced if for all $l \to r \in R$, r is normal in R and l is normal in $R \backslash \{l \to r\}$.

$$aab \to aba; \quad baa \to aba$$

is not terminating, since string $aaba$ gives birth to the rewrite sequence $aaba \to abaa \to aaba \to \ldots$. The SRS $ab \to bbaa$ isn't terminating either, given context aab.

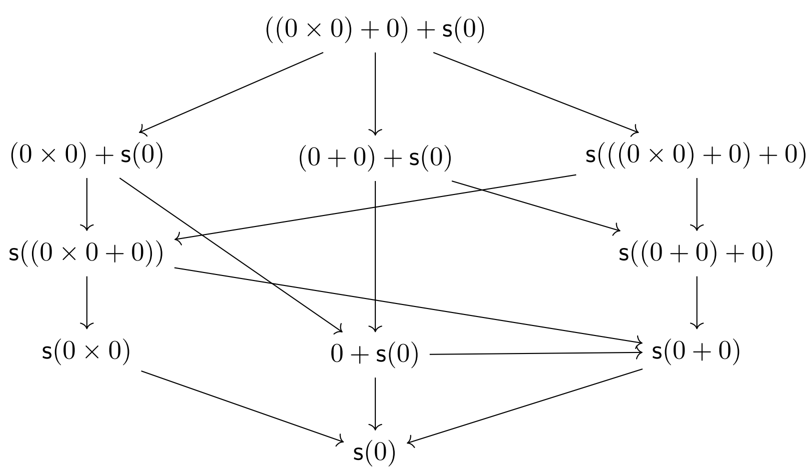
Exercise 2 :

Given the following term rewriting system (TRS):

$$\begin{aligned} x \times \mathsf{0} &\to \mathsf{0} & \quad x + \mathsf{0} &\to x \\ \mathsf{0} \times x &\to \mathsf{0} & \mathsf{0} + x &\to x \\ \mathsf{s}(x) \times y &\to (x \times y) + y & x + \mathsf{s}(y) &\to \mathsf{s}(x + y) \\ x \times \mathsf{s}(y) &\to (x \times y) + x & \mathsf{s}(x) + y &\to \mathsf{s}(x + y) \end{aligned}$$

Show the *reduction graph* of $((0 \times 0) + 0) + \mathsf{s}(0)$.

Solution:



Exercise 3 :
Given the signature $(\{\mathbb{N},\ \mathsf{List}\}, \{\mathsf{0},\ \mathsf{s},\ \epsilon,\ :,\ \mathbin{/\!\!\backslash\!\!\backslash},\ \mathsf{sort}\})$ where the set of functions is typed as follows:

$$\mathsf{0} : \mathbb{N}, \qquad \mathsf{s} : \mathbb{N} \to \mathbb{N}, \qquad \epsilon : \mathsf{List}, \qquad (:) : \mathbb{N} \times \mathsf{List} \to \mathsf{List},$$
$$\mathbin{/\!\!\backslash\!\!\backslash} : \mathsf{List} \times \mathsf{List} \to \mathsf{List}, \qquad \mathsf{sort} : \mathsf{List} \to \mathsf{List}$$

Define a finite TRS that simulates the *mergesort algorithm*. If needed, you can define auxiliary sorts and function symbols.

Solution:

We will use the additional sort $\mathbb{B} = \{\top, \bot\}$ and the following function symbols:

$\mathsf{even} : \mathsf{List} \to \mathsf{List}$, $\mathsf{odd} : \mathsf{List} \to \mathsf{List}$, $\geq : \mathbb{N} \times \mathbb{N} \to \mathbb{B}$, $\mathsf{aux} : \mathbb{N} \times \mathsf{List} \times \mathsf{List} \to \mathsf{List}$

We define the following TRS:

$$\mathsf{even}(\epsilon) \to \epsilon \qquad \mathsf{odd}(\epsilon) \to \epsilon$$
$$\mathsf{even}(x{:}\epsilon) \to \epsilon \qquad \mathsf{odd}(x{:}\epsilon) \to x{:}\epsilon$$
$$\mathsf{even}(x{:}y{:}z) \to y{:}\mathsf{even}(z) \qquad \mathsf{odd}(x{:}y{:}z) \to x{:}\mathsf{odd}(z)$$

$$\mathsf{0} \geq \mathsf{0} \to \top \qquad \mathsf{aux}(\top, x{:}y, z{:}w) \to z{:} \mathbin{/\!\!\backslash\!\!\backslash} (x{:}y, w)$$
$$\mathsf{s}(x) \geq \mathsf{0} \to \top \qquad \mathsf{aux}(\bot, x{:}y, z{:}w) \to x{:} \mathbin{/\!\!\backslash\!\!\backslash} (y, z{:}w)$$
$$\mathsf{0} \geq \mathsf{s}(x) \to \bot$$
$$\mathsf{s}(x) \geq \mathsf{s}(y) \to x \geq y$$

$$\mathbin{/\!\!\backslash\!\!\backslash}(x, \epsilon) \to x$$
$$\mathbin{/\!\!\backslash\!\!\backslash}(\epsilon, x) \to x$$
$$\mathbin{/\!\!\backslash\!\!\backslash}(x{:}y, z{:}w) \to \mathsf{aux}(x \geq z, x{:}y, z{:}w)$$

$$\mathsf{sort}(\epsilon) \to \epsilon$$
$$\mathsf{sort}(x{:}\epsilon) \to x{:}\epsilon$$
$$\mathsf{sort}(x{:}y{:}z) \to \mathbin{/\!\!\backslash\!\!\backslash}(\mathsf{sort}(\mathsf{even}(x{:}y{:}z)), \mathsf{sort}(\mathsf{odd}(x{:}y{:}z)))$$

A **polynomial interpretation on integers** is the following:

- a subset A of $\mathbb{N}$;
- for every symbol f of arity n, a polynomial $\mathsf{P}_f \in \mathbb{N}[X_1, \ldots, X_n]$;
- for every $a_1, \ldots, a_n \in A$, $\mathsf{P}_f(a_1, \ldots, a_n) \in A$;
- for every $a_1, \ldots, a_i > a'_i, \ldots, a_n \in A$, $\mathsf{P}_f(a_1, \ldots, a_i, \ldots, a_n) > \mathsf{P}_f(a_1, \ldots, a'_i, \ldots, a_n)$;

Then $(A, (\mathsf{P}_f)_f, >)$ is a well-founded monotone algebra.

Exercise 4 :
Consider the TRS R consisting of the rewrite rules

$$\mathsf{0} + y \to y \qquad \mathsf{0} \dot{-} y \to \mathsf{0} \qquad \mathsf{min}(x, y) \to x \dot{-} (x \dot{-} y)$$
$$\mathsf{s}(x) + y \to \mathsf{s}(x + y) \qquad \mathsf{s}(x) \dot{-} \mathsf{0} \to \mathsf{s}(x) \qquad \mathsf{max}(x, y) \to (x + y) \dot{-} \mathsf{min}(x, y)$$
$$\mathsf{s}(x) \dot{-} \mathsf{s}(y) \to x \dot{-} y$$

(a) Show that the following interpretation over $\mathbb{N}$ is indeed a interpretation and that is compatible with R (that is, for each rule $\ell \to r \in R$, we have, for any assignment α, $[\alpha](\ell) >_{\mathbb{N}} [\alpha](r)$)

$$\mathsf{0}_{\mathbb{N}} = 0 \qquad +_{\mathbb{N}}(x, y) = 2x + y + 1 \qquad \mathsf{min}(x, y) = 2x + y + 3$$
$$\mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad \dot{-}_{\mathbb{N}}(x, y) = x + y + 1 \qquad \mathsf{max}_{\mathbb{N}}(x, y) = 4x + 2y + 6$$

Solution:

$[](\mathsf{0} + y) = y + 1 > y = [](y) \quad [](\mathsf{s}(x) + y) = 2x + y + 3 > 2x + y + 2 = [](\mathsf{s}(x + y))$

(b) Find natural numbers a, b, c, d, e and f such that the following interpretation over $\mathbb{N}$ is compatible with R,

$$\begin{array}{lll} \mathsf{0}_{\mathbb{N}} = 0 & +_{\mathbb{N}}(x,y) = 2x + y + 1 & \mathsf{min}_{\mathbb{N}}(x,y) = 3x + by + c \\ \mathsf{s}_{\mathbb{N}}(x) = x + 1 & \dot{-}_{\mathbb{N}}(x,y) = x + 2y + a & \mathsf{max}_{\mathbb{N}}(x,y) = dx + ey + f \end{array}$$

> **Solution:**
> The compatibility condition provides the following inequalities
>
> $$a > 0$$
> $$(b-4)y + c - 3a > 0$$
> $$(d-8)x + (e-2b-1)y + f - 2c - a - 1 > 0$$
>
> which are satisified, for all $(x, y) \in (\mathbb{N} \times \mathbb{N})$, with values $(a, b, c, d, e, f) = (1, 4, 4, 8, 10, 11)$.

Exercise 5 :
Prove the termination of the following TRS

$$\mathsf{0} \times x \to \mathsf{0} \qquad\qquad x + \mathsf{0} \to x$$
$$\mathsf{s}(x) \times y \to (x \times y) + y \qquad\qquad x + \mathsf{s}(y) \to \mathsf{s}(x + y)$$

using the polynomial interpretation on natural numbers:

$$\mathsf{P}_0 = 2 \qquad \mathsf{P}_{\mathsf{s}}(X) = X + 1 \qquad \mathsf{P}_+(X,Y) = X + 2Y \qquad \mathsf{P}_\times(X,Y) = (X+Y)^2$$

> **Solution:**
> From the polynomial interpretation we get the following polynomial for the various rules of the TRS: $\mathsf{P}_{0\times x}(X) = (X+2)^2$, $\mathsf{P}_{\mathsf{s}(x)\times y}(X,Y) = (X+Y+1)^2$, $\mathsf{P}_{(x\times y)+y}(X,Y) = (X+Y)^2 + 2Y$, $\mathsf{P}_{x+0}(X) = X + 4$, $\mathsf{P}_{x+\mathsf{s}(y)}(X,Y) = X + 2(Y+1)$ and $\mathsf{P}_{\mathsf{s}(x+y)} = X + 2Y + 1$.
>
> - $\mathsf{P}_{0\times x}(X) > \mathsf{P}_0$ true since $(X+2)^2 = X^2 + 4X + 4 > 2$;
> - $\mathsf{P}_{\mathsf{s}(x)\times y}(X,Y) > \mathsf{P}_{(x\times y)+y}(X,Y)$ true since $(X+Y+1)^2 = X^2+2XY+Y^2+2X+2Y+1$ is greater than $(X+Y)^2 + 2Y = X^2 + 2XY + Y^2 + 2Y$;
> - $\mathsf{P}_{x+0}(X) > X$ true since $X + 4 > X$;
> - $\mathsf{P}_{x+\mathsf{s}(y)}(X,Y) > \mathsf{P}_{\mathsf{s}(x+y)}$ since $X + 2(Y+1) > X + 2Y + 1$.

Is this polynomial interpretation suitable to prove termination of the TRS of Exercise 2?

> **Solution:**
> No. For the rule $\mathsf{s}(x) + y \to \mathsf{s}(x+y)$. Indeed, $\mathsf{P}_{\mathsf{s}(x)+y}(X,Y) = \mathsf{P}_{\mathsf{s}(x+y)}(X,Y) = X + 2Y + 1$.

Exercise 6 :
Let R be a rewrite system on a signature $\mathcal{F}$, and I a model of R, that is, an $\mathcal{F}$-algebra $(A, (f_I)_{f\in\mathcal{F}})$ such that $R \subseteq =_I$ where $t =_I u$ iff for all $\xi : \mathcal{V} \to A$, $t\xi = u\xi$.

Let $\mathcal{F}^I$ be the signature such that $\mathsf{f}_{a_1,\ldots,a_n} \in \mathcal{F}^I_n$ iff $\mathsf{f} \in \mathcal{F}_n$ and let $\mathsf{lab}(R) = \{\mathsf{lab}(\ell,\xi) \to \mathsf{lab}(r,\xi) | \ell \to r \in R, \xi : \mathcal{V} \to A\}$, where $\mathsf{lab}(x,\xi) = x$ and $\mathsf{lab}(\mathsf{f}\ t_1\ \ldots\ t_n, \xi) = \mathsf{f}_{t_1\xi,\ldots,t_n\xi}\ \mathsf{lab}(t_1,\xi)\ \ldots\ \mathsf{lab}(t_n,\xi)$, where for any term t, $t\xi \in A$ is the substitution generalised to terms that is, the rewrite system obtained by labeling function symbols by the semantics of their arguments.

1. Prove that $\to_R$ terminates iff $\to_{\mathsf{lab}(R)}$ terminates.

Solution:

- $\Rightarrow$ Assume there is an infinite rewrite sequence $t_1 \to_{\mathsf{lab}(R)} t_2 \ldots$. Then by stripping the labels, we obtain an infinite rewrite sequence in R, $t_1 \to_R t_2 \ldots$; which is impossible since R is terminating.

- $\Leftarrow$ Assume there is an infinite rewrite sequence $t_1 \to_R t_2 \ldots$. We will show that this sequence gives birth to an infinite rewrite sequence in $\mathsf{lab}(R)$. The sequence can be written out with contexts C_i and substitutions σ_i,

$$C_i[\ell_i\sigma_i] \to_R C_i[r_i\sigma_i] = C_{i+1}[\ell_{i+1}\sigma_{i+1}] \to_R C_{i+1}[r_{i+1}\sigma_{i+1}]$$

  with for all i, $\ell_i \to r_i \in R$.

  We now show that each rewrite step $C_i[\ell_i\sigma_i] \to C_i[r_i\sigma_i]$ in R gives rise to a $\mathsf{lab}(R)$ rewrite step.

  - Note that, $\forall\sigma,\xi$, $\mathsf{lab}(t\sigma,\xi) = \mathsf{lab}(t,\xi\circ\sigma)\mathsf{lab}(\sigma,\xi)$ where for any substitution σ, if $\sigma(x) = t$, then $\mathsf{lab}(\sigma,\xi)(x) = \mathsf{lab}(t,\xi)$ (can be proved by induction on the structure of terms).
    Therefore,

$$\mathsf{lab}(\ell\sigma,\xi) = \mathsf{lab}(\ell,\xi\circ\sigma)\mathsf{lab}(\sigma,\xi) \to_{\mathsf{lab}(R)} \mathsf{lab}(r,\xi\circ\sigma)\mathsf{lab}(\sigma,\xi) = \mathsf{lab}(r\sigma,\xi) \quad (1)$$

    The rewriting is allowed because

    - if $\ell \to r \in R$, then $\forall\xi, \mathsf{lab}(\ell,\xi) \to_{\mathsf{lab}(R)} \mathsf{lab}(r,\xi)$ (and $\xi\circ\sigma$ is a valid valuation);
    - the rewriting relation is closed by substitution, and $\mathsf{lab}(\ell,\xi\circ\sigma) \to_{\mathsf{lab}(R)} \mathsf{lab}(r,\xi\circ\sigma)$ and $\mathsf{lab}(\sigma,\xi)$ is a substitution.

  - We now show that lab is compatible with $\mathcal{F}^I$ operations to be able to build contexts. Assume $\ell \to r \in R$, $t_1,\ldots t_{m-1}, t_{m+1},\ldots t_n \in \mathcal{F}$ and $f \in \mathcal{F}_n$. Then

$$\mathsf{lab}\,(f\ t_1\ \cdots\ \ell\ \cdots\ t_n) = f_{t_1\xi\ \cdots\ \ell\xi\ \cdots t_n\xi}\ \mathsf{lab}(t_1,\xi)\ \cdots\ \mathsf{lab}(\ell,\xi)\ \cdots\ \mathsf{lab}(t_n,\xi) \to_{\mathsf{lab}(R)} f_{t_1\xi\ \cdots\ r\xi\ \cdots\ t_n\xi}\ \mathsf{lab}(t_1,\xi)\ \cdots\ \mathsf{lab}(r,\xi)\ \cdots\ \mathsf{lab}(t_n,\xi) = \mathsf{lab}(f\ t_1\ \cdots\ r\ \cdots\ t_n,\xi) \quad (2)$$

    with the rewriting allowed since $\ell\xi =_I r\xi$ because $R \subseteq =_I$.

  And since $A \neq \emptyset$ by definition of an $\mathcal{F}$ algebra, a valuation ξ can always be found and a sequence in $\mathsf{lab}(R)$ can always be built.

2. Prove that a polynomial interpretation cannot prove the termination of the following system

$$\mathsf{f}\ (\mathsf{s}\ X) \to \mathsf{f}\ (\mathsf{p}\ (\mathsf{s}\ X)) \diamond (\mathsf{s}\ X) \qquad \mathsf{p}\ (\mathsf{s}\ \mathsf{z}) \to \mathsf{z}$$
$$\mathsf{p}\ (\mathsf{s}\ (\mathsf{s}\ X)) \to \mathsf{s}\ (\mathsf{p}\ (\mathsf{s}\ X))$$

Solution:
Because we are on $\mathbb{N}$, we have that $P_{\mathsf{p}} \geq id_{\mathbb{N}}$ and $P_\diamond(\cdot,y) \geq id_{\mathbb{N}}$. Consequently,

$$P_\diamond((P_{\mathsf{f}} \circ P_{\mathsf{p}} \circ P_{\mathsf{s}})(X), P_{\mathsf{s}}(X)) \geq (P_{\mathsf{f}} \circ P_{\mathsf{s}})(X)$$

3. Prove that this rewrite system can be proved terminating using 1.

Solution:
Use as model $\mathbb{N}$ where $s\ n = n+1$, $p\ n = n \dot{-} 1$, $\diamond\ n\ m = n+m$ and $f\ n = \frac{n*(n+1)}{2}$.
Take

- $P_{\mathsf{f}_i}(X) = (2^{i+1}-1)X + i$,
- $P_{\mathsf{p}}(X) = 2X$,
- $P_{\mathsf{s}}(X) = X + 1$,
- $P_{\mathsf{z}} = 0$ and
- $P_{\diamond}(X, Y) = X + Y$.

and compute

- $P_{\diamond}(P_{\mathsf{f}_i} \circ P_{\mathsf{p}} \circ P_{\mathsf{s}}(X), P_{\mathsf{s}}(X)) = (2^{i+1}-1) * 2(X+1) + i + X + 1$
$= (2^{i+2} - 2 + 1)X + 2^{i+2} - 2 + i + 1$
$= (2^{i+2} - 1)(X+1) + i$
which is smaller than $P_{\mathsf{f}_{i+1}} \circ P_{\mathsf{s}}(X) = (2^{i+2}-1)(X+1) + i + 1$;
- $P_{\mathsf{p}} \circ P_{\mathsf{s}} \circ P_{\mathsf{z}} = 2$ which is greater than $P_{\mathsf{z}} = 0$;
- $P_{\mathsf{p}} \circ P_{\mathsf{s}} \circ P_{\mathsf{s}} = 2X + 4$ greater than $P_{\mathsf{s}} \circ P_{\mathsf{p}} \circ P_{\mathsf{s}} = 2X + 3$.

Additionnally, one has to verify that each polynomial is (strictly) increasing for all its variables.

A **polynomial interpretation on real numbers** is the following:

- a subset A of $\mathbb{R}^+$;
- a positive real number δ;
- for every symbol f of arity n, a polynomial $\mathsf{P}_f \in \mathbb{R}[X_1, \ldots, X_n]$;
- for every $a_1, \ldots, a_n \in A$, $\mathsf{P}_f(a_1, \ldots, a_n) \in A$;
- for every $a_1, \ldots, a_i >_\delta a'_i, \ldots, a_n \in A$, $\mathsf{P}_f(a_1, \ldots, a_i, \ldots, a_n) >_\delta \mathsf{P}_f(a_1, \ldots, a'_i, \ldots, a_n)$ where $x >_\delta y$ iff $x > y + \delta$.

Then $(A, (\mathsf{P}_f)_f, >_\delta)$ is a well-founded monotone algebra.

Exercise 7 :
Consider the following two TRS:

$$\begin{aligned} R_1 = \{ &\ \mathsf{l}(\mathsf{p}(x)) \to \mathsf{p}(\mathsf{p}(\mathsf{l}(x))),\ \mathsf{p}(\mathsf{s}(x)) \to \mathsf{s}(\mathsf{s}(\mathsf{p}(x))),\ \mathsf{p}(x) \to \mathsf{a}(x,x), \\ &\ \mathsf{s}(x) \to \mathsf{a}(x, \mathsf{0}),\ \mathsf{s}(x) \to \mathsf{a}(\mathsf{0}, x)\ \} \\ R_2 = \{ &\ \mathsf{r}(\mathsf{r}(\mathsf{r}(x))) \to \mathsf{a}(\mathsf{r}(x), \mathsf{r}(x)),\ \mathsf{s}(\mathsf{a}(\mathsf{r}(x), \mathsf{r}(x))) \to \mathsf{r}(\mathsf{r}(\mathsf{r}(x)))\ \} \end{aligned}$$

1. Prove that $R_1 \cup R_2$ terminates using the following polynomial interpretation on real numbers: $\delta = 1$, $\mathsf{P}_{\mathsf{0}}(X) = 0$, $\mathsf{P}_{\mathsf{l}}(X) = X^2$, $\mathsf{P}_{\mathsf{s}}(X) = X + 4$, $\mathsf{P}_{\mathsf{p}}(X) = 3X + 5$, $\mathsf{P}_{\mathsf{a}}(X, Y) = X + Y$ and $\mathsf{P}_{\mathsf{r}}(X) = \sqrt{2}X + 1$.

Solution:

$$\begin{aligned} \mathsf{P}_{\mathsf{l}(\mathsf{p}(x))}(X) = 9X^2 + 30X + 25 &>_1 \mathsf{P}_{\mathsf{p}(\mathsf{p}(\mathsf{l}(x)))}(X) = 9X^2 + 20 \\ \mathsf{P}_{\mathsf{p}(\mathsf{s}(x))}(X) = 3X + 17 &>_1 \mathsf{P}_{\mathsf{s}(\mathsf{s}(\mathsf{p}(x)))}(X) = 3X + 13 \\ \mathsf{P}_{\mathsf{p}(x)}(X) = 3X + 5 &>_1 \mathsf{P}_{\mathsf{a}(x,x)}(X) = 2X \\ \mathsf{P}_{\mathsf{s}(x)}(X) = X + 4 &>_1 \mathsf{P}_{\mathsf{a}(x,\mathsf{0})}(X) = X \\ \mathsf{P}_{\mathsf{s}(x)}(X) = X + 4 &>_1 \mathsf{P}_{\mathsf{a}(\mathsf{0},x)} = X \\ \mathsf{P}_{\mathsf{r}(\mathsf{r}(\mathsf{r}(x)))}(X) = 2\sqrt{2}X + 3 + \sqrt{2} &>_1 \mathsf{P}_{\mathsf{a}(\mathsf{r}(x),\mathsf{r}(x))} = 2\sqrt{2}X + 2 \\ \mathsf{P}_{\mathsf{s}(\mathsf{a}(\mathsf{r}(x),\mathsf{r}(x)))}(X) = 2\sqrt{2}X + 6 &>_1 \mathsf{P}_{\mathsf{r}(\mathsf{r}(\mathsf{r}(x)))}(X) = 2\sqrt{2}X + 3 + \sqrt{2} \end{aligned}$$

2. Prove that in any polynomial interpretation on natural numbers proving the termination of R_1 it must hold that $\mathsf{P}_{\mathsf{s}}(X)$ is of the form $X + s_0$ and $\mathsf{P}_{\mathsf{a}}(X, Y)$ is of the form $X + Y + a_0$, with $s_0 > a_0$.

   hint: look at the dominant terms of the polynomials computed from the rewrite rules.

   > **Solution:**
   > Let $P_0 = z \geq 0$. From the second rule of R_1, let α be the degree of $\mathsf{P}_{\mathsf{s}}(X)$ and let β be the degree of $\mathsf{P}_{\mathsf{p}}(X)$. From $\mathsf{P}_{\mathsf{ps}(x)}(X) > \mathsf{P}_{\mathsf{s}(\mathsf{s}(\mathsf{p}(x)))}(X)$ it must hold that $\beta\alpha \geq \alpha\alpha\beta$. Therefore $\alpha = 1$. Similarly, from the first rule, also $\mathsf{P}_{\mathsf{p}}(X)$ is of degree one. From the third rule it must hold that $\mathsf{P}_{\mathsf{a}}(X, Y)$ is also of degree one. So $\mathsf{P}_{\mathsf{p}}(X)$ is of the form $p_1X + p_0$, $\mathsf{P}_{\mathsf{s}}(X)$ is of the form $s_1X + s_0$ whereas $\mathsf{P}_{\mathsf{a}}(X, Y)$ is of the form $a_2X + a_1Y + a_0$. From the fourth rule it must hold $s_1X + s_0 > a_2X + a_0 + a_1z$, which implies $s_1 \geq a_2 \geq 1$. Similarly, from the fifth rule, $s_1 \geq a_1 \geq 1$. From the second rule $s_1p_1X + s_0p_1 + p_0 > s_1^2p_1X + s_1^2p_0 + s_1s_0 + s_0$ and therefore it must hold that $s_1p_1 \geq s_1^2p_1$. Therefore $s_1 = 1$, which also implies $a_2 = a_1 = 1$. Moreover from $s_1X + s_0 > a_2X + a_0 + a_1z$, it must hold $s_0 > a_0$.

3. Deduce that the termination of $R_1 \cup R_2$ cannot be proved using a polynomial interpretation of integers.

   > **Solution:**
   > Let α be the degree of the polynomial $\mathsf{P}_{\mathsf{r}}(X)$. From the second rule of R_2 it must hold that $\alpha^3 \leq \alpha$ and therefore $\alpha = 1$ and $\mathsf{P}_{\mathsf{r}}(X)$ is of the form $r_1X + r_0$. Looking now at the first rule, it must hold that $r_1(r_1(r_1X + r_0) + r_0) + r_0 > 2r_1X + 2r_0 + a_0$ which implies $r_1^3 \geq 2r_1$ and therefore $r_1^2 \geq 2$. Similarly, from the second rule of R_2 it must hold that $2r_1 \geq r_1^3$ or alternatively $r_1^2 \leq 2$. Therefore r_1^2 must be equal to 2, which requires r_1 $(= \sqrt{2})$ not to be a natural number.