

Rewriting techniques, I: basics, interpretation, termination

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Exercise 1 :

(a) Given the reduction rules

$$((s\ x) + y) \triangleright (s\ (x + y)); \quad (0 + x) \triangleright x$$

Can $(s\ ((s\ 0) + 0))$ be *reduced*? Can it be *rewritten*? Provide the substitution, the context and the term t being reduced.

(b) A *string rewrite system* (SRS for short) is a TRS over a signature that contains only unary function symbols. Given the (string) reductions

$$a(b(x)) \triangleright b(a(x))$$

Can $a(a(b(x)))$ be reduced? Can $a(b(a(x)))$ be reduced? Can they be rewritten?

(c) Build a reduced string rewrite system that is not terminating.

Exercise 2 :

Given the following term rewriting system (TRS):

$$\begin{array}{ll} x \times 0 \rightarrow 0 & x + 0 \rightarrow x \\ 0 \times x \rightarrow 0 & 0 + x \rightarrow x \\ s(x) \times y \rightarrow (x \times y) + y & x + s(y) \rightarrow s(x + y) \\ x \times s(y) \rightarrow (x \times y) + x & s(x) + y \rightarrow s(x + y) \end{array}$$

Show the *reduction graph* of $((0 \times 0) + 0) + s(0)$.

Exercise 3 :

Given the signature $(\{\mathbb{N}, \text{List}\}, \{0, s, \epsilon, :, \mathbb{M}, \text{sort}\})$ where the set of functions is typed as follows:

$$\begin{array}{llll} 0 : \mathbb{N}, & s : \mathbb{N} \rightarrow \mathbb{N}, & \epsilon : \text{List}, & (:) : \mathbb{N} \times \text{List} \rightarrow \text{List}, \\ \mathbb{M} : \text{List} \times \text{List} \rightarrow \text{List}, & \text{sort} : \text{List} \rightarrow \text{List} & & \end{array}$$

Define a finite TRS that simulates the *mergesort algorithm*. If needed, you can define auxiliary sorts and function symbols.

A **polynomial interpretation on integers** is the following:

- a subset A of \mathbb{N} ;
- for every symbol f of arity n , a polynomial $P_f \in \mathbb{N}[X_1, \dots, X_n]$;
- for every $a_1, \dots, a_n \in A$, $P_f(a_1, \dots, a_n) \in A$;
- for every $a_1, \dots, a_i > a'_i, \dots, a_n \in A$, $P_f(a_1, \dots, a_i, \dots, a_n) > P_f(a_1, \dots, a'_i, \dots, a_n)$;

Then $(A, (P_f)_f, >)$ is a well-founded monotone algebra.

Exercise 4 :

Consider the TRS R consisting of the rewrite rules

$$\begin{array}{lll} 0 + y \rightarrow y & 0 \dot{-} y \rightarrow 0 & \min(x, y) \rightarrow x \dot{-} (x \dot{-} y) \\ s(x) + y \rightarrow s(x + y) & s(x) \dot{-} 0 \rightarrow s(x) & \max(x, y) \rightarrow (x + y) \dot{-} \min(x, y) \\ & s(x) \dot{-} s(y) \rightarrow x \dot{-} y & \end{array}$$

(a) Show that the following interpretation over \mathbb{N} is indeed a interpretation and that is compatible with R (that is, for each rule $\ell \rightarrow r \in R$, we have, for any assignment α , $[\alpha](\ell) >_{\mathbb{N}} [\alpha](r)$)

$$\begin{array}{lll} 0_{\mathbb{N}} = 0 & +_{\mathbb{N}}(x, y) = 2x + y + 1 & \min(x, y) = 2x + y + 3 \\ s_{\mathbb{N}}(x) = x + 1 & \dot{-}_{\mathbb{N}}(x, y) = x + y + 1 & \max_{\mathbb{N}}(x, y) = 4x + 2y + 6 \end{array}$$

- (b) Find natural numbers a, b, c, d, e and f such that the following interpretation over \mathbb{N} is compatible with R ,

$$\begin{array}{lll} 0_{\mathbb{N}} = 0 & +_{\mathbb{N}}(x, y) = 2x + y + 1 & \min_{\mathbb{N}}(x, y) = 3x + by + c \\ s_{\mathbb{N}}(x) = x + 1 & -_{\mathbb{N}}(x, y) = x + 2y + a & \max_{\mathbb{N}}(x, y) = dx + ey + f \end{array}$$

Exercise 5 :

Prove the termination of the following TRS

$$\begin{array}{ll} 0 \times x \rightarrow 0 & x + 0 \rightarrow x \\ s(x) \times y \rightarrow (x \times y) + y & x + s(y) \rightarrow s(x + y) \end{array}$$

using the polynomial interpretation on natural numbers:

$$P_0 = 2 \quad P_s(X) = X + 1 \quad P_+(X, Y) = X + 2Y \quad P_{\times}(X, Y) = (X + Y)^2$$

Is this polynomial interpretation suitable to prove termination of the TRS of Exercise 2?

Exercise 6 :

Let R be a rewrite system on a signature \mathcal{F} , and I a model of R , that is, an \mathcal{F} -algebra $(A, (f_I)_{f \in \mathcal{F}})$ such that $R \subseteq =_I$ where $t =_I u$ iff for all $\xi : \mathcal{V} \rightarrow A$, $t\xi = u\xi$.

Let \mathcal{F}^I be the signature such that $f_{a_1, \dots, a_n} \in \mathcal{F}_n^I$ iff $f \in \mathcal{F}_n$ and let $\text{lab}(R) = \{\text{lab}(\ell, \xi) \rightarrow \text{lab}(r, \xi) \mid \ell \rightarrow r \in R, \xi : \mathcal{V} \rightarrow A\}$, where $\text{lab}(x, \xi) = x$ and $\text{lab}(f \ t_1 \ \dots \ t_n, \xi) = f_{t_1\xi, \dots, t_n\xi} \ \text{lab}(t_1, \xi) \ \dots \ \text{lab}(t_n, \xi)$, where for any term t , $t\xi \in A$ is the substitution generalised to terms that is, the rewrite system obtained by labeling function symbols by the semantics of their arguments.

1. Prove that \rightarrow_R terminates iff $\rightarrow_{\text{lab}(R)}$ terminates.
2. Prove that a polynomial interpretation cannot prove the termination of the following system

$$\begin{array}{ll} f \ (s \ X) \rightarrow f \ (p \ (s \ X)) \diamond (s \ X) & p \ (s \ z) \rightarrow z \\ p \ (s \ (s \ X)) \rightarrow s \ (p \ (s \ X)) & \end{array}$$

3. Prove that this rewrite system can be proved terminating using 1.

A **polynomial interpretation on real numbers** is the following:

- a subset A of \mathbb{R}^+ ;
- a positive real number δ ;
- for every symbol f of arity n , a polynomial $P_f \in \mathbb{R}[X_1, \dots, X_n]$;
- for every $a_1, \dots, a_n \in A$, $P_f(a_1, \dots, a_n) \in A$;
- for every $a_1, \dots, a_i >_{\delta} a'_i, \dots, a_n \in A$, $P_f(a_1, \dots, a_i, \dots, a_n) >_{\delta} P_f(a_1, \dots, a'_i, \dots, a_n)$ where $x >_{\delta} y$ iff $x > y + \delta$.

Then $(A, (P_f)_f, >_{\delta})$ is a well-founded monotone algebra.

Exercise 7 :

Consider the following two TRS:

$$\begin{array}{l} R_1 = \{ l(p(x)) \rightarrow p(p(l(x))), \ p(s(x)) \rightarrow s(s(p(x))), \ p(x) \rightarrow a(x, x), \\ \quad s(x) \rightarrow a(x, 0), \ s(x) \rightarrow a(0, x) \} \\ R_2 = \{ r(r(r(x))) \rightarrow a(r(x), r(x)), \ s(a(r(x), r(x))) \rightarrow r(r(r(x))) \} \end{array}$$

1. Prove that $R_1 \cup R_2$ terminates using the following polynomial interpretation on real numbers: $\delta = 1$, $P_0(X) = 0$, $P_l(X) = X^2$, $P_s(X) = X + 4$, $P_p(X) = 3X + 5$, $P_a(X, Y) = X + Y$ and $P_r(X) = \sqrt{2}X + 1$.
2. Prove that in any polynomial interpretation on natural numbers proving the termination of R_1 it must hold that $P_s(X)$ is of the form $X + s_0$ and $P_a(X, Y)$ is of the form $X + Y + a_0$, with $s_0 > a_0$.
hint: look at the dominant terms of the polynomials computed from the rewrite rules.
3. Deduce that the termination of $R_1 \cup R_2$ cannot be proved using a polynomial interpretation of integers.